6.1 | Graphs of the Sine and Cosine Functions

Learning Objectives

In this section, you will:

6.1.1 Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
6.1.2 Use phase shifts of sine and cosine curves.

Figure 6.2  Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/Flickr)

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the $x$- and $y$-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let’s start with the sine function. We can create a table of values and use them to sketch a graph. Table 6.1 lists some of the values for the sine function on a unit circle.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(x)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.1

Plotting the points from the table and continuing along the $x$-axis gives the shape of the sine function. See Figure 6.3.
Notice how the sine values are positive between 0 and $\pi$, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between $\pi$ and $2\pi$, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure 6.4.

Now let’s take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. Table 6.2 lists some of the values for the cosine function on a unit circle.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\pi/6$</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
<th>$2\pi/3$</th>
<th>$3\pi/4$</th>
<th>$5\pi/6$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cos($x$)</td>
<td>1</td>
<td>$\sqrt{3}/2$</td>
<td>$\sqrt{2}/2$</td>
<td>$1/2$</td>
<td>0</td>
<td>$-1/2$</td>
<td>$-\sqrt{2}/2$</td>
<td>$-\sqrt{3}/2$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Table 6.2

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 6.5.

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after $2\pi$, which means the functions are periodic with a period of $2\pi$. A periodic function is a function for which a specific horizontal shift, $P$, results in a function equal to the original function:
\[ f(x + P) = f(x) \] for all values of \( x \) in the domain of \( f \). When this occurs, we call the smallest such horizontal shift with \( P > 0 \) the period of the function. Figure 6.6 shows several periods of the sine and cosine functions.

![Figure 6.6](image)

Looking again at the sine and cosine functions on a domain centered at the \( y \)-axis helps reveal symmetries. As we can see in Figure 6.7, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because \( \sin(-x) = -\sin x \). Now we can clearly see this property from the graph.

![Figure 6.7](image)

Figure 6.7  Odd symmetry of the sine function

Figure 6.8 shows that the cosine function is symmetric about the \( y \)-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that \( \cos(-x) = \cos x \).

![Figure 6.8](image)

Figure 6.8  Even symmetry of the cosine function

**Characteristics of Sine and Cosine Functions**

The sine and cosine functions have several distinct characteristics:
• They are periodic functions with a period of $2\pi$.
• The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
• The graph of $y = \sin x$ is symmetric about the origin, because it is an odd function.
• The graph of $y = \cos x$ is symmetric about the $y$-axis, because it is an even function.

**Investigating Sinusoidal Functions**

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

\[
y = A\sin(Bx - C) + D
\]

and

\[
y = A\cos(Bx - C) + D
\]

**Determining the Period of Sinusoidal Functions**

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, $B$ is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than $2\pi$ and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than $2\pi$ and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is $2\pi$, which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is $\pi$ and the graph is compressed. If $f(x) = \sin(\frac{x}{2})$, then $B = \frac{1}{2}$, so the period is $4\pi$ and the graph is stretched. Notice in Figure 6.9 how the period is indirectly related to $|B|$.  

![Figure 6.9](image)

**Period of Sinusoidal Functions**

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

\[
y = A\sin(Bx)
\]

\[
y = A\cos(Bx)
\]

The period is $\frac{2\pi}{|B|}$.

**Example 6.1**
### Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{x}{6}\right)$.

**Solution**

Let’s begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot 6 = 12$$

Determine the period of the function $g(x) = \cos\left(\frac{x}{3}\right)$.

### Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable $B$ relates to the period. Now let’s turn to the variable $A$ so we can analyze how it is related to the amplitude, or greatest distance from rest. $A$ represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical midline of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the $x$-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4\sin x$ is twice the amplitude of $f(x) = 2\sin x$. If $|A| < 1$, the function is compressed. Figure 6.10 compares several sine functions with different amplitudes.

![Figure 6.10: Amplitude of Sinusoidal Functions](http://legacy.cnx.org/content/col11667/1.4)

**Amplitude of Sinusoidal Functions**

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms
Example 6.2

Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function \( f(x) = -4\sin(x) \)? Is the function stretched or compressed vertically?

Solution

Let’s begin by comparing the function to the simplified form \( y = A\sin(Bx) \).

In the given function, \( A = -4 \), so the amplitude is \( |A| = |-4| = 4 \). The function is stretched.

Analysis

The negative value of \( A \) results in a reflection across the \( x \)-axis of the sine function, as shown in Figure 6.11.

![Figure 6.11](image.png)

Try it 6.2  What is the amplitude of the sinusoidal function \( f(x) = \frac{1}{2}\sin(x) \)? Is the function stretched or compressed vertically?

Analyzing Graphs of Variations of \( y = \sin x \) and \( y = \cos x \)

Now that we understand how \( A \) and \( B \) relate to the general form equation for the sine and cosine functions, we will explore the variables \( C \) and \( D \). Recall the general form:

\[
y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D
\]

or

\[
y = A\sin(B\left(x - \frac{C}{B}\right)) + D \quad \text{and} \quad y = A\cos(B\left(x - \frac{C}{B}\right)) + D
\]
The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. **Figure 6.12** shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by $\pi$ units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

![Figure 6.12](image)

While $C$ relates to the horizontal shift, $D$ indicates the vertical shift from the midline in the general formula for a sinusoidal function. See **Figure 6.13**. The function $y = \cos(x) + D$ has its midline at $y = D$.

![Figure 6.13](image)

Any value of $D$ other than zero shifts the graph up or down. **Figure 6.14** compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

![Figure 6.14](image)

**Variations of Sine and Cosine Functions**

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and $D$ is the vertical shift.
Example 6.3

**Identifying the Phase Shift of a Function**

Determine the direction and magnitude of the phase shift for \( f(x) = \sin(x + \frac{\pi}{6}) - 2 \).

**Solution**

Let’s begin by comparing the equation to the general form \( y = A\sin(Bx - C) + D \).

In the given equation, notice that \( B = 1 \) and \( C = -\frac{\pi}{6} \). So the phase shift is

\[
\frac{C}{B} = -\frac{\pi}{6}
\]

or \( \frac{\pi}{6} \) units to the left.

**Analysis**

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before \( C \). Therefore \( f(x) = \sin(x + \frac{\pi}{6}) - 2 \) can be rewritten as \( f(x) = \sin(\left(x - \left(-\frac{\pi}{6}\right)\right)) - 2 \).

If the value of \( C \) is negative, the shift is to the left.

Try It 6.3 Determine the direction and magnitude of the phase shift for \( f(x) = 3\cos(x - \frac{\pi}{2}) \).

Example 6.4

**Identifying the Vertical Shift of a Function**

Determine the direction and magnitude of the vertical shift for \( f(x) = \cos(x) - 3 \).

**Solution**

Let’s begin by comparing the equation to the general form \( y = A\cos(Bx - C) + D \).

In the given equation, \( D = -3 \) so the shift is 3 units downward.

Try It 6.4 Determine the direction and magnitude of the vertical shift for \( f(x) = 3\sin(x) + 2 \).
Given a sinusoidal function in the form \( f(x) = A\sin(Bx - C) + D \), identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as \(|A|\).
2. Determine the period as \( P = \frac{2\pi}{|B|} \).
3. Determine the phase shift as \( \frac{C}{B} \).
4. Determine the midline as \( y = D \).

**Example 6.5**

**Identifying the Variations of a Sinusoidal Function from an Equation**

Determine the midline, amplitude, period, and phase shift of the function \( y = 3\sin(2x) + 1 \).

**Solution**

Let’s begin by comparing the equation to the general form \( y = A\sin(Bx - C) + D \).

\( A = 3 \), so the amplitude is \(|A| = 3\).

Next, \( B = 2 \), so the period is \( P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi \).

There is no added constant inside the parentheses, so \( C = 0 \) and the phase shift is \( \frac{C}{B} = \frac{0}{2} = 0 \).

Finally, \( D = 1 \), so the midline is \( y = 1 \).

**Analysis**

Inspecting the graph, we can determine that the period is \( \pi \), the midline is \( y = 1 \), and the amplitude is 3. See Figure 6.15.

**Figure 6.15**

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**Example 6.6**

Determine the midline, amplitude, period, and phase shift of the function \( y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right) \).
Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure 6.16.

![Figure 6.16](image)

**Solution**

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

\[
y = A\sin(Bx - C) + D
\]

\[
y = A\cos(Bx - C) + D
\]

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When \( x = 0 \), the graph has an extreme point, \((0, 0)\). Since the cosine function has an extreme point for \( x = 0 \), let us write our equation in terms of a cosine function.

Let’s start with the midline. We can see that the graph rises and falls an equal distance above and below \( y = 0.5 \). This value, which is the midline, is \( D \) in the equation, so \( D = 0.5 \).

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So \( |A| = 0.5 \). Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so \( |A| = \frac{1}{2} = 0.5 \). Also, the graph is reflected about the \( x \)-axis so that \( A = -0.5 \).

The graph is not horizontally stretched or compressed, so \( B = 1 \); and the graph is not shifted horizontally, so \( C = 0 \).

Putting this all together,

\[
g(x) = -0.5\cos(x) + 0.5
\]

**Example 6.7**

Determine the formula for the sine function in Figure 6.17.

![Figure 6.17](image)
Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 6.18.

**Solution**
With the highest value at 1 and the lowest value at \(-5\), the midline will be halfway between at \(-2\). So \(D = -2\).

The distance from the midline to the highest or lowest value gives an amplitude of \(|A| = 3\).

The period of the graph is 6, which can be measured from the peak at \(x = 1\) to the next peak at \(x = 7\), or from the distance between the lowest points. Therefore, \(P = \frac{2\pi}{|B|} = 6\). Using the positive value for \(B\), we find that

\[
B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}
\]

So far, our equation is either \(y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2\) or \(y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2\). For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes
6.7 Write a formula for the function graphed in Figure 6.19.

Graphing Variations of $y = \sin x$ and $y = \cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D,$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

**Given the function** $y = A\sin(Bx)$, **sketch its graph.**

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if $A$ is positive or decreasing if $A$ is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the $x$-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the $x$-axis at $x = \frac{\pi}{2|B|}$.

Again, these functions are equivalent, so both yield the same graph.
Example 6.8

Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of \( f(x) = -2\sin\left(\frac{\pi x}{2}\right) \).

Solution

Let’s begin by comparing the equation to the form \( y = A\sin(Bx) \).

Step 1. We can see from the equation that \( A = -2 \), so the amplitude is 2.

\[ |A| = 2 \]

Step 2. The equation shows that \( B = \frac{\pi}{2} \), so the period is

\[ P = \frac{2\pi}{B} \]
\[ = \frac{2\pi}{\frac{\pi}{2}} \]
\[ = 4 \]

Step 3. Because \( A \) is negative, the graph descends as we move to the right of the origin.

Step 4–7. The \( x \)-intercepts are at the beginning of one period, \( x = 0 \), the horizontal midpoints are at \( x = 2 \) and at the end of one period at \( x = 4 \).

The quarter points include the minimum at \( x = 1 \) and the maximum at \( x = 3 \). A local minimum will occur 2 units below the midline, at \( x = 1 \), and a local maximum will occur at 2 units above the midline, at \( x = 3 \).

Figure 6.20 shows the graph of the function.

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8. Sketch a graph of \( g(x) = -0.8\cos(2x) \). Determine the midline, amplitude, period, and phase shift.
Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph.

1. Express the function in the general form \( y = A \sin(Bx - C) + D \) or \( y = A \cos(Bx - C) + D \).
2. Identify the amplitude, \(|A|\).
3. Identify the period, \( P = \frac{2\pi}{|B|} \).
4. Identify the phase shift, \( \frac{C}{B} \).
5. Draw the graph of \( f(x) = A \sin(Bx) \) shifted to the right or left by \( \frac{C}{B} \) and up or down by \( D \).

Example 6.9

Graphing a Transformed Sinusoid

Sketch a graph of \( f(x) = 3 \sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right) \).

Solution

Step 1. The function is already written in general form: \( f(x) = 3 \sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right) \). This graph will have the shape of a sine function, starting at the midline and increasing to the right.

Step 2. \(|A| = |3| = 3\). The amplitude is 3.

Step 3. Since \(|B| = \left|\frac{\pi}{4}\right| = \frac{\pi}{4}\), we determine the period as follows.

\[
P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8
\]

The period is 8.

Step 4. Since \( C = \frac{\pi}{4} \), the phase shift is

\[
\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1.
\]

The phase shift is 1 unit.

Step 5. Figure 6.21 shows the graph of the function.
Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Example 6.10

**Identifying the Properties of a Sinusoidal Function**

Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

**Solution**

Begin by comparing the equation to the general form and use the steps outlined in Example 6.9.

\[ y = A\cos(Bx - C) + D \]

**Step 1.** The function is already written in general form.

**Step 2.** Since $A = -2$, the amplitude is $|A| = 2$.

**Step 3.** $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.

**Step 4.** $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = \frac{-\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is $-2$.

**Step 5.** $D = 3$, so the midline is $y = 3$, and the vertical shift is up 3.

Since $A$ is negative, the graph of the cosine function has been reflected about the $x$-axis.

**Figure 6.22** shows one cycle of the graph of the function.
Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

Example 6.11

Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the $y$-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius $r$, the $y$-coordinate of the point is $y = r \sin(x)$, so in this case, we get the equation $y(x) = 3 \sin(x)$. The constant 3 causes a vertical stretch of the $y$-values of the function by a factor of 3, which we can see in the graph in Figure 6.23.

Analysis
Notice that the period of the function is still $2\pi$; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi, \ldots$. Because the outputs of the graph will now oscillate between $-3$ and $3$, the amplitude of the sine wave is $3$.

6.10 What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Example 6.12

**Finding the Vertical Component of Circular Motion**

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled $P$, as shown in Figure 6.24. Sketch a graph of the height above the ground of the point $P$ as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

![Figure 6.24](http://legacy.cnx.org/content/col11667/1.4)

**Solution**

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 6.25.
Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let’s use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$
6.11 A weight is attached to a spring that is then hung from a board, as shown in Figure 6.26. As the spring oscillates up and down, the position $y$ of the weight relative to the board ranges from $-1$ in. (at time $x = 0$) to $-7$ in. (at time $x = \pi$) below the board. Assume the position of $y$ is given as a sinusoidal function of $x$. Sketch a graph of the function, and then find a cosine function that gives the position $y$ in terms of $x$.

![Figure 6.26](image)

**Example 6.13**

**Determining a Rider’s Height on a Ferris Wheel**

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider’s height above ground as a function of time in minutes.

**Solution**

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider’s height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$
where $t$ is in minutes and $y$ is measured in meters.

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine (http://openstaxcollege.org/l/ampperiod)
- Translations of Sine and Cosine (http://openstaxcollege.org/l/translasincos)
- Graphing Sine and Cosine Transformations (http://openstaxcollege.org/l/transformsincos)
- Graphing the Sine Function (http://openstaxcollege.org/l/graphsinefunc)
6.1 EXERCISES

Verbal

1. Why are the sine and cosine functions called periodic functions?

2. How does the graph of \( y = \sin x \) compare with the graph of \( y = \cos x \)? Explain how you could horizontally translate the graph of \( y = \sin x \) to obtain \( y = \cos x \).

3. For the equation \( A \cos(Bx + C) + D \), what constants affect the range of the function and how do they affect the range?

4. How does the range of a translated sine function relate to the equation \( y = A \sin(Bx + C) + D \)?

5. How can the unit circle be used to construct the graph of \( f(t) = \sin t \)?

Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum \( y \)-values and their corresponding \( x \)-values on one period for \( x > 0 \). Round answers to two decimal places if necessary.

6. \( f(x) = 2 \sin x \)

7. \( f(x) = \frac{2}{3} \cos x \)

8. \( f(x) = -3 \sin x \)

9. \( f(x) = 4 \sin x \)

10. \( f(x) = 2 \cos x \)

11. \( f(x) = \cos(2x) \)

12. \( f(x) = 2 \sin \left( \frac{1}{2}x \right) \)

13. \( f(x) = 4 \cos(2\pi x) \)

14. \( f(x) = 3 \cos \left( \frac{6}{5}x \right) \)

15. \( y = 3 \sin(8(x + 4)) + 5 \)

16. \( y = 2 \sin(3x - 21) + 4 \)

17. \( y = 5 \sin(5x + 20) - 2 \)

For the following exercises, graph one full period of each function, starting at \( x = 0 \). For each function, state the amplitude, period, and midline. State the maximum and minimum \( y \)-values and their corresponding \( x \)-values on one period for \( x > 0 \). State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. \( f(t) = 2 \sin(t - \frac{5\pi}{6}) \)

19. \( f(t) = -\cos \left( t + \frac{\pi}{3} \right) + 1 \)
20. \[ f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3 \]

21. \[ f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right) \]

22. \[ f(x) = 4\sin\left(\frac{\pi}{2}(x - 3)\right) + 7 \]

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure 6.27.

![Figure 6.27](image)

24. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.28.

![Figure 6.28](image)

25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.29.

![Figure 6.29](image)
26. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in Figure 6.30.

27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.31.

28. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in Figure 6.32.

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.33.
30. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in Figure 6.34.

**Algebraic**

For the following exercises, let \( f(x) = \sin x \).

31. On \([0, 2\pi]\), solve \( f(x) = 0 \).

32. On \([0, 2\pi]\), solve \( f(x) = \frac{1}{2} \).

33. Evaluate \( f\left(\frac{\pi}{2}\right) \).

34. On \([0, 2\pi]\), \( f(x) = \frac{\sqrt{2}}{2} \). Find all values of \( x \).

35. On \([0, 2\pi]\), the maximum value(s) of the function occur(s) at what \( x \)-value(s)?

36. On \([0, 2\pi]\), the minimum value(s) of the function occur(s) at what \( x \)-value(s)?

37. Show that \( f(-x) = -f(x) \). This means that \( f(x) = \sin x \) is an odd function and possesses symmetry with respect to ____________.

For the following exercises, let \( f(x) = \cos x \).

38. On \([0, 2\pi]\), solve the equation \( f(x) = \cos x = 0 \).

39. On \([0, 2\pi]\), solve \( f(x) = \frac{1}{2} \).

40. On \([0, 2\pi]\), find the \( x \)-intercepts of \( f(x) = \cos x \).

41. On \([0, 2\pi]\), find the \( x \)-values at which the function has a maximum or minimum value.

42. On \([0, 2\pi]\), solve the equation \( f(x) = \frac{\sqrt{3}}{2} \).
Technology

43. Graph \( h(x) = x + \sin x \) on \([0, 2\pi]\). Explain why the graph appears as it does.

44. Graph \( h(x) = x + \sin x \) on \([-100, 100]\). Did the graph appear as predicted in the previous exercise?

45. Graph \( f(x) = x \sin x \) on \([0, 2\pi]\) and verbalize how the graph varies from the graph of \( f(x) = \sin x \).

46. Graph \( f(x) = x \sin x \) on the window \([-10, 10]\) and explain what the graph shows.

47. Graph \( f(x) = \frac{\sin x}{x} \) on the window \([-5\pi, 5\pi]\) and explain what the graph shows.

Real-World Applications

48. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o’clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function \( h(t) \) gives a person’s height in meters above the ground \( t \) minutes after the wheel begins to turn.

   a. Find the amplitude, midline, and period of \( h(t) \).

   b. Find a formula for the height function \( h(t) \).

   c. How high off the ground is a person after 5 minutes?
6.2 | Graphs of the Other Trigonometric Functions

**Learning Objectives**

In this section, you will:

- 6.2.1 Analyze the graph of \( y = \tan x \).
- 6.2.2 Graph variations of \( y = \tan x \).
- 6.2.3 Analyze the graphs of \( y = \sec x \) and \( y = \csc x \).
- 6.2.4 Graph variations of \( y = \sec x \) and \( y = \csc x \).
- 6.2.5 Analyze the graph of \( y = \cot x \).
- 6.2.6 Graph variations of \( y = \cot x \).

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

**Analyzing the Graph of \( y = \tan x \)**

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

\[
\tan x = \frac{\sin x}{\cos x}
\]

The period of the tangent function is \( \pi \) because the graph repeats itself on intervals of \( k\pi \) where \( k \) is a constant. If we graph the tangent function on \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\), we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

\[
\tan(-x) = \frac{\sin(-x)}{\cos(-x)}
\]

Definition of tangent.

\[
= \frac{-\sin x}{\cos x}
\]

Sine is an odd function, cosine is even.

\[
= -\frac{\sin x}{\cos x}
\]

The quotient of an odd and an even function is odd.

\[
= -\tan x
\]

Definition of tangent.

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 6.3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\frac{\pi}{2})</th>
<th>(-\frac{\pi}{3})</th>
<th>(-\frac{\pi}{4})</th>
<th>(-\frac{\pi}{6})</th>
<th>0</th>
<th>(\frac{\pi}{6})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{3})</th>
<th>(\frac{\pi}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan(x) )</td>
<td>undefined</td>
<td>(-\sqrt{3})</td>
<td>(-1)</td>
<td>(-\sqrt{3})</td>
<td>0</td>
<td>(\sqrt{3})</td>
<td>1</td>
<td>(\sqrt{3})</td>
<td>undefined</td>
</tr>
</tbody>
</table>

Table 6.3

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when \( \frac{\pi}{3} < x < \frac{\pi}{2} \), we can use a table to look for a trend. Because \( \frac{\pi}{3} \approx 1.05 \) and \( \frac{\pi}{2} \approx 1.57 \), we will evaluate \( x \) at radian measures \( 1.05 < x < 1.57 \) as shown in Table 6.4.
As $x$ approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 6.5.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.3</th>
<th>1.5</th>
<th>1.55</th>
<th>1.56</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan x$</td>
<td>3.6</td>
<td>14.1</td>
<td>48.1</td>
<td>92.6</td>
</tr>
</tbody>
</table>

Table 6.4

We can see that, as $x$ approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of $x$ for which $\cos x = 0$. For example, $\cos \left(\frac{\pi}{2}\right) = 0$ and $\cos \left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes.

Figure 6.35 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from $\pi$ to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the tangent function can be described by a general equation.

$y = Atan(Bx)$

We can identify horizontal and vertical stretches and compressions using values of $A$ and $B$. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term amplitude cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase stretching/compressing factor when referring to the constant $A$. 
### Features of the Graph of \( y = \text{Atan}(Bx) \)

- The stretching factor is \(|A|\).
- The period is \( P = \frac{\pi}{|B|} \).
- The domain is all real numbers \( x \), where \( x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k \) such that \( k \) is an integer.
- The range is \((-\infty, \infty)\).
- The asymptotes occur at \( x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k \), where \( k \) is an integer.
- \( y = \text{Atan}(Bx) \) is an odd function.

### Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the tangent function to quickly sketch a graph of any stretched and/or compressed tangent function of the form \( f(x) = \text{Atan}(Bx) \). We focus on a single period of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function’s domain if we wish. Our limited domain is then the interval \( \left(-\frac{P}{2}, \frac{P}{2}\right) \) and the graph has vertical asymptotes at \( \pm \frac{P}{2} \) where \( P = \frac{\pi}{B} \). On \( \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \), the graph will come up from the left asymptote at \( x = -\frac{\pi}{2} \), cross through the origin, and continue to increase as it approaches the right asymptote at \( x = \frac{\pi}{2} \). To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

\[
f\left(\frac{P}{4}\right) = \text{Atan}\left(B\frac{P}{4}\right) = \text{Atan}\left(B\frac{\pi}{4B}\right) = A
\]

because \( \tan\left(\frac{\pi}{4}\right) = 1 \).

**Given the function** \( f(x) = \text{Atan}(Bx) \), **graph one period.**

1. Identify the stretching factor, \(|A|\).
2. Identify \( B \) and determine the period, \( P = \frac{\pi}{|B|} \).
3. Draw vertical asymptotes at \( x = -\frac{P}{2} \) and \( x = \frac{P}{2} \).
4. For \( A > 0 \), the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for \( A < 0 \)).
5. Plot reference points at \( \left(\frac{P}{4}, A\right) \), \((0, 0)\), and \( \left(-\frac{P}{4}, -A\right) \), and draw the graph through these points.

### Example 6.14

**Sketching a Compressed Tangent**

Sketch a graph of one period of the function \( y = 0.5 \text{tan}\left(\frac{\pi}{2}x\right) \).
Solution
First, we identify $A$ and $B$.

$$y = 0.5 \tan(\frac{\pi}{2} x)$$

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$f(0.5) = 0.5 \tan\left(\frac{0.5\pi}{2}\right) = 0.5 \tan\left(\frac{\pi}{4}\right) = 0.5$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$, and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure 6.36 shows the graph of one period of the function.

Figure 6.36

6.12 Sketch a graph of $f(x) = 3 \tan\left(\frac{\pi}{6} x\right)$.

Graphing One Period of a Shifted Tangent Function
Now that we can graph a tangent function that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add $C$ and $D$ to the general form of the tangent function.

$$f(x) = A \tan(Bx - C) + D$$  \hspace{1cm} (6.3)

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

**Features of the Graph of $y = A \tan(Bx - C) + D$**

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
• The domain is \( x \neq \frac{C}{B} + \frac{\pi}{2|B|}k \), where \( k \) is an integer.

• The range is \((-\infty, -|A|] \cup [|A|, \infty)\).

• The vertical asymptotes occur at \( x = \frac{C}{B} + \frac{\pi}{2|B|}k \), where \( k \) is an odd integer.

• There is no amplitude.

• \( y = A \tan(Bx) \) is an odd function because it is the quotient of odd and even functions (sin and cosine, respectively).

Given the function \( y = A \tan(Bx - C) + D \), sketch the graph of one period.

1. Express the function given in the form \( y = A \tan(Bx - C) + D \).
2. Identify the stretching/compressing factor, \( |A| \).
3. Identify \( B \) and determine the period, \( P = \frac{\pi}{|B|} \).
4. Identify \( C \) and determine the phase shift, \( \frac{C}{B} \).
5. Draw the graph of \( y = A \tan(Bx) \) shifted to the right by \( \frac{C}{B} \) and up by \( D \).
6. Sketch the vertical asymptotes, which occur at \( x = \frac{C}{B} + \frac{\pi}{2|B|}k \), where \( k \) is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 6.15

Graphing One Period of a Shifted Tangent Function

Graph one period of the function \( y = -2\tan(\pi x + \pi) - 1 \).

Solution

Step 1. The function is already written in the form \( y = A \tan(Bx - C) + D \).

Step 2. \( A = -2 \), so the stretching factor is \( |A| = 2 \).

Step 3. \( B = \pi \), so the period is \( P = \frac{\pi}{|B|} = \frac{\pi}{\pi} = 1 \).

Step 4. \( C = -\pi \), so the phase shift is \( \frac{C}{B} = \frac{-\pi}{\pi} = -1 \).

Step 5-7. The asymptotes are at \( x = -\frac{3}{2} \) and \( x = -\frac{1}{2} \) and the three recommended reference points are \((-1.25, 1), (-1,-1), \) and \((-0.75,-3)\). The graph is shown in Figure 6.37.
6.13 How would the graph in Example 6.15 look different if we made $A = 2$ instead of $-2$?

Analysis
Note that this is a decreasing function because $A < 0$.

**Example 6.16**

**Identifying the Graph of a Stretched Tangent**

Find a formula for the function graphed in Figure 6.38.

**Solution**
The graph has the shape of a tangent function.

*Step 1.* One cycle extends from $-4$ to $4$, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{2\pi}{8} = \frac{\pi}{4}$. 

**How To:**
Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period $P$ from the spacing between successive vertical asymptotes or $x$-intercepts.
2. Write $f(x) = \tan \left( \frac{\pi}{P} x \right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine $A$. 

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Step 2. The equation must have the form \( f(x) = \tan\left(\frac{\pi}{8}x\right) \).

Step 3. To find the vertical stretch \( A \), we can use the point \((2, 2)\).

\[
2 = A \tan\left(\frac{\pi}{8} \cdot 2\right) = A \tan\left(\frac{\pi}{4}\right)
\]

Because \( \tan\left(\frac{\pi}{4}\right) = 1 \), \( A = 2 \).

This function would have a formula \( f(x) = 2\tan\left(\frac{\pi}{8}x\right) \).

Try 6.14 Find a formula for the function in Figure 6.39.

![Figure 6.39](image)

Analyzing the Graphs of \( y = \sec x \) and \( y = \csc x \)

The secant was defined by the reciprocal identity \( \sec x = \frac{1}{\cos x} \). Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at \( \frac{\pi}{2}, \frac{3\pi}{2} \), etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph \( y = \sec x \) by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 6.40. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of \( x \) where the cosine graph crosses the \( x \)-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, \( \sec(-x) = \sec x \).
Figure 6.40  Graph of the secant function, 
\[ f(x) = \sec x = \frac{1}{\cos x} \]

As we did for the tangent function, we will again refer to the constant \( |A| \) as the stretching factor, not the amplitude.

### Features of the Graph of \( y = A \sec(Bx) \)

- The stretching factor is \( |A| \).
- The period is \( \frac{2\pi}{|B|} \).
- The domain is \( x \neq \frac{\pi}{2|B|} k \), where \( k \) is an odd integer.
- The range is \( (-\infty, -|A|] \cup [|A|, \infty) \).
- The vertical asymptotes occur at \( x = \frac{\pi}{2|B|} k \), where \( k \) is an odd integer.
- There is no amplitude.
- \( y = A \sec(Bx) \) is an even function because cosine is an even function.

Similar to the secant, the cosecant is defined by the reciprocal identity  \( \csc x = \frac{1}{\sin x} \). Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, \( \pi \), etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph \( y = \csc x \) by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure 6.41. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the cosecant function decreases.

The cosecant graph has vertical asymptotes at each value of \( x \) where the sine graph crosses the \( x \)-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, \( \csc(-x) = -\csc x \).

The graph of cosecant, which is shown in Figure 6.41, is similar to the graph of secant.
Figure 6.41 The graph of the cosecant function,
\[ f(x) = \csc x = \frac{1}{\sin x} \]

**Features of the Graph of** \( y = Acsc(Bx) \)

- The stretching factor is \( |A| \).
- The period is \( \frac{2\pi}{|B|} \).
- The domain is \( x \neq \frac{\pi}{|B|} k \), where \( k \) is an integer.
- The range is \( (-\infty, -|A|] \cup [|A|, \infty) \).
- The asymptotes occur at \( x = \frac{\pi}{|B|} k \), where \( k \) is an integer.
- \( y = Acsc(Bx) \) is an odd function because sine is an odd function.

**Graphing Variations of** \( y = \sec x \) and \( y = \csc x \)

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions. The equations become the following.

\[
\begin{align*}
y &= Asec(Bx - C) + D \\
y &= Acsc(Bx - C) + D
\end{align*}
\]

(6.4) \hfill (6.5)

**Features of the Graph of** \( y = Asec(Bx-C)+D \)

- The stretching factor is \( |A| \).
- The period is \( \frac{2\pi}{|B|} \).
- The domain is \( x \neq \frac{C}{B} + \frac{\pi}{2|B|} k \), where \( k \) is an odd integer.
• The range is \((-\infty, -|A|] \cup [|A|, \infty)\).

• The vertical asymptotes occur at \(x = \frac{C}{B} + \frac{\pi}{2|B|}k\), where \(k\) is an odd integer.

• There is no amplitude.

• \(y = A\sec(Bx)\) is an even function because cosine is an even function.

### Features of the Graph of \(y = A\csc(Bx-C)+D\)

- The stretching factor is \(|A|\).
- The period is \(\frac{2\pi}{|B|}\).
- The domain is \(x \neq \frac{C}{B} + \frac{\pi}{2|B|}k\), where \(k\) is an integer.
- The range is \((-\infty, -|A|] \cup [|A|, \infty)\).
- The vertical asymptotes occur at \(x = \frac{C}{B} + \frac{\pi}{2|B|}k\), where \(k\) is an integer.
- There is no amplitude.
- \(y = A\csc(Bx)\) is an odd function because sine is an odd function.

### How To: Given a function of the form \(y = A\sec(Bx)\), graph one period.

1. Express the function given in the form \(y = A\sec(Bx)\).
2. Identify the stretching/compressing factor, \(|A|\).
3. Identify \(B\) and determine the period, \(P = \frac{2\pi}{|B|}\).
4. Sketch the graph of \(y = A\cos(Bx)\).
5. Use the reciprocal relationship between \(y = \cos x\) and \(y = \sec x\) to draw the graph of \(y = A\sec(Bx)\).
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

### Example 6.17

#### Graphing a Variation of the Secant Function

Graph one period of \(f(x) = 2.5\sec(0.4x)\).

**Solution**

*Step 1.* The given function is already written in the general form, \(y = A\sec(Bx)\).

*Step 2.* \(A = 2.5\) so the stretching factor is 2.5.
Step 3. B = 0.4 so \( P = \frac{2\pi}{0.4} = 5\pi \). The period is 5\( \pi \) units.

Step 4. Sketch the graph of the function \( g(x) = 2.5\cos(0.4x) \).

Step 5. Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.

Steps 6–7. Sketch two asymptotes at \( x = 1.25\pi \) and \( x = 3.75\pi \). We can use two reference points, the local minimum at (0, 2.5) and the local maximum at (2.5\( \pi \), −2.5). Figure 6.42 shows the graph.

![Figure 6.42](image)

**Try It** Graph one period of \( f(x) = -2.5\sec(0.4x) \).

**Do the vertical shift and stretch/compression affect the secant's range?**

Yes. The range of \( f(x) = A\sec(Bx - C) + D \) is \((-\infty, \ -|A| + D] \cup [|A| + D, \infty)\).
Given a function of the form \( f(x) = A \sec(Bx - C) + D \), graph one period.

1. Express the function given in the form \( y = A \sec(Bx - C) + D \).
2. Identify the stretching/compressing factor, \( |A| \).
3. Identify \( B \) and determine the period, \( \frac{2\pi}{|B|} \).
4. Identify \( C \) and determine the phase shift, \( \frac{C}{B} \).
5. Draw the graph of \( y = A \sec(Bx) \), but shift it to the right by \( \frac{C}{B} \) and up by \( D \).
6. Sketch the vertical asymptotes, which occur at \( x = \frac{C}{B} + \frac{\pi}{2|B|}k \), where \( k \) is an odd integer.

Example 6.18

Graphing a Variation of the Secant Function

Graph one period of \( y = 4\sec\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 1 \).

Solution

Step 1. Express the function given in the form \( y = 4\sec\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 1 \).

Step 2. The stretching/compressing factor is \( |A| = 4 \).

Step 3. The period is

\[
\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{3}} = 6
\]

Step 4. The phase shift is

\[
\frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{3}} = \frac{3\pi}{2\pi} = 1.5
\]

Step 5. Draw the graph of \( y = A\sec(Bx) \), but shift it to the right by \( \frac{C}{B} = 1.5 \) and up by \( D = 6 \).

Step 6. Sketch the vertical asymptotes, which occur at \( x = 0, x = 3 \), and \( x = 6 \). There is a local minimum at \((1.5, 5)\) and a local maximum at \((4.5, 7)\). Figure 6.43 shows the graph.
Graph one period of \( f(x) = -6 \sec(4x + 2) - 8 \).

The domain of \( \csc x \) was given to be all \( x \) such that \( x \neq k\pi \) for any integer \( k \). Would the domain of \( y = A \csc(Bx - C) + D \) be \( x \neq C + \frac{k\pi}{B} \)?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function’s input.

Given a function of the form \( y = A \csc(Bx) \), graph one period.

1. Express the function given in the form \( y = A \csc(Bx) \).
2. \(|A|\).
3. Identify \( B \) and determine the period, \( P = \frac{2\pi}{|B|} \).
4. Draw the graph of \( y = \sin(Bx) \).
5. Use the reciprocal relationship between \( y = \sin x \) and \( y = \csc x \) to draw the graph of \( y = A \csc(Bx) \).
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 6.19

Graphing a Variation of the Cosecant Function

Graph one period of \( f(x) = -3 \csc(4x) \).

Solution
Step 1. The given function is already written in the general form, \( y = A \csc(Bx) \).

Step 2. \(|A| = |-3| = 3\), so the stretching factor is 3.

Step 3. \( B = 4 \), so \( P = \frac{2\pi}{4} = \frac{\pi}{2} \). The period is \( \frac{\pi}{2} \) units.

Step 4. Sketch the graph of the function \( g(x) = -3\sin(4x) \).

Step 5. Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

Steps 6–7. Sketch three asymptotes at \( x = 0 \), \( x = \frac{\pi}{4} \), and \( x = \frac{\pi}{2} \). We can use two reference points, the local maximum at \( \left( \frac{\pi}{8}, -3 \right) \) and the local minimum at \( \left( \frac{3\pi}{8}, 3 \right) \). Figure 6.44 shows the graph.

\[ \text{Figure 6.44} \]

6.17 Graph one period of \( f(x) = 0.5\csc(2x) \).
Given a function of the form \( f(x) = A \csc(Bx - C) + D \), graph one period.

1. Express the function given in the form \( y = A \csc(Bx - C) + D \).
2. Identify the stretching/compressing factor, \(|A|\).
3. Identify \( B \) and determine the period, \( \frac{2\pi}{|B|} \).
4. Identify \( C \) and determine the phase shift, \( \frac{C}{B} \).
5. Draw the graph of \( y = A \csc(Bx) \) but shift it to the right by and up by \( D \).
6. Sketch the vertical asymptotes, which occur at \( x = \frac{C}{B} + \frac{\pi}{|B|} k \), where \( k \) is an integer.

### Example 6.20

**Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant**

Sketch a graph of \( y = 2\csc\left(\frac{\pi}{2}x\right) + 1 \). What are the domain and range of this function?

**Solution**

*Step 1.* Express the function given in the form \( y = 2\csc\left(\frac{\pi}{2}x\right) + 1 \).

*Step 2.* Identify the stretching/compressing factor, \(|A| = 2\).

*Step 3.* The period is \( \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1} \cdot \frac{2}{\pi} = 4 \).

*Step 4.* The phase shift is \( \frac{0}{2} = 0 \).

*Step 5.* Draw the graph of \( y = A \csc(Bx) \) but shift it up \( D = 1 \).

*Step 6.* Sketch the vertical asymptotes, which occur at \( x = 0, x = 2, x = 4 \).

The graph for this function is shown in Figure 6.45.
Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of \( f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1 \), shown as the orange dashed wave.

**Try it** 6.18 Given the graph of \( f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1 \) shown in Figure 6.46, sketch the graph of \( g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1 \) on the same axes.

**Analyzing the Graph of** \( y = \cot x \)

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity \( \cot x = \frac{1}{\tan x} \). Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the
graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph \( y = \cot x \) by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 6.47. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of \( x \) where \( \tan x = 0 \); we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, \( \cot x \) has vertical asymptotes at all values of \( x \) where \( \tan x = 0 \), and \( \cot x = 0 \) at all values of \( x \) where \( \tan x \) has its vertical asymptotes.

![Figure 6.47](image.png)

**Figure 6.47** The cotangent function

### Features of the Graph of \( y = A \cot(Bx) \)

- The stretching factor is \( |A| \).
- The period is \( P = \frac{\pi}{|B|} \).
- The domain is \( x \neq \frac{\pi}{|B|} k \), where \( k \) is an integer.
- The range is \( (-\infty, \infty) \).
- The asymptotes occur at \( x = \frac{\pi}{|B|} k \), where \( k \) is an integer.
- \( y = A \cot(Bx) \) is an odd function.

### Graphing Variations of \( y = \cot x \)

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

\[
y = A \cot(Bx - C) + D
\]

(6.6)
Properties of the Graph of \( y = \text{Acot}(Bx-C)+D \)

- The stretching factor is \(|A|\).
- The period is \( \frac{\pi}{|B|} \).
- The domain is \( x \neq \frac{C}{B} + \frac{\pi}{|B|}k \), where \( k \) is an integer.
- The range is \( (-\infty, -|A|] \cup [|A|, \infty) \).
- The vertical asymptotes occur at \( x = \frac{C}{B} + \frac{\pi}{|B|}k \), where \( k \) is an integer.
- There is no amplitude.
- \( y = \text{Acot}(Bx) \) is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively).

**Given a modified cotangent function of the form \( f(x) = \text{Acot}(Bx) \), graph one period.**

1. Express the function in the form \( f(x) = \text{Acot}(Bx) \).
2. Identify the stretching factor, \(|A|\).
3. Identify the period, \( P = \frac{\pi}{|B|} \).
4. Draw the graph of \( y = \text{Atan}(Bx) \).
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of \( y = \text{Acot}(Bx) \).
7. Sketch the asymptotes.

**Example 6.21**

**Graphing Variations of the Cotangent Function**

Determine the stretching factor, period, and phase shift of \( y = 3\text{cot}(4x) \), and then sketch a graph.

**Solution**

*Step 1.* Expressing the function in the form \( f(x) = \text{Acot}(Bx) \) gives \( f(x) = 3\text{cot}(4x) \).

*Step 2.* The stretching factor is \(|A| = 3\).

*Step 3.* The period is \( P = \frac{\pi}{4} \).

*Step 4.* Sketch the graph of \( y = 3\text{tan}(4x) \).

*Step 5.* Plot two reference points. Two such points are \( \left( \frac{\pi}{16}, 3 \right) \) and \( \left( \frac{3\pi}{16}, -3 \right) \).

*Step 6.* Use the reciprocal relationship to draw \( y = 3\text{cot}(4x) \).

*Step 7.* Sketch the asymptotes, \( x = 0, \ x = \frac{\pi}{4} \).
The orange graph in Figure 6.48 shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by $D$.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where $k$ is an integer.
7. Plot any three reference points and draw the graph through these points.
Sketch a graph of one period of the function \( f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2. \)

**Solution**

*Step 1.* The function is already written in the general form \( f(x) = A\cot(Bx - C) + D. \)

*Step 2.* \( A = 4, \) so the stretching factor is 4.

*Step 3.* \( B = \frac{\pi}{8}, \) so the period is \( P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8. \)

*Step 4.* \( C = \frac{\pi}{2}, \) so the phase shift is \( \frac{C}{B} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4. \)

*Step 5.* We draw \( f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2. \)

*Step 6-7.* Three points we can use to guide the graph are \((6, 2), (8, -2),\) and \((10, -6).\) We use the reciprocal relationship of tangent and cotangent to draw \( f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2. \)

*Step 8.* The vertical asymptotes are \( x = 4 \) and \( x = 12.\)

The graph is shown in Figure 6.49.

**Using the Graphs of Trigonometric Functions to Solve Real-World Problems**

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let’s return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the tangent function.

**Example 6.23**

**Using Trigonometric Functions to Solve Real-World Scenarios**
Suppose the function \( y = 5\tan\left(\frac{\pi}{4}t\right) \) marks the distance in the movement of a light beam from the top of a police car across a wall where \( t \) is the time in seconds and \( y \) is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.

b. Graph on the interval \([0, 5]\).

c. Evaluate \( f(1) \) and discuss the function’s value at that input.

**Solution**

a. We know from the general form of \( y = A\tan(Bt) \) that \(|A|\) is the stretching factor and \( \frac{\pi}{B} \) is the period.

\[ y = 5 \tan\left(\frac{\pi}{4}t\right) \]

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is \( \frac{\pi}{\frac{\pi}{4}} = 4 \). This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at \( t = 2 \) and use the stretching factor and period. See Figure 6.51

\[ f(1) = 5\tan\left(\frac{\pi}{4}(1)\right) = 5(1) = 5; \] after 1 second, the beam of light has moved 5 ft from the spot across from the police car.

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent (http://openstaxcollege.org/l/graphtangent)
- Graphing Cosecant and Secant (http://openstaxcollege.org/l/graphcscsec)
- Graphing the Cotangent (http://openstaxcollege.org/l/graphcot)
6.2 EXERCISES

Verbal

49. Explain how the graph of the sine function can be used to graph $y = \csc x$.

50. How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

51. Explain why the period of $\tan x$ is equal to $\pi$.

52. Why are there no intercepts on the graph of $y = \csc x$?

53. How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

Algebraic

For the following exercises, match each trigonometric function with one of the graphs in Figure 6.52.

Figure 6.52

54. $f(x) = \tan x$

55. $f(x) = \sec x$

56. $f(x) = \csc x$

57. $f(x) = \cot x$
For the following exercises, find the period and horizontal shift of each of the functions.

58. \( f(x) = 2\tan(4x - 32) \)

59. \( h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right) \)

60. \( m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right) \)

61. If \( \tan x = -1.5 \), find \( \tan(-x) \).

62. If \( \sec x = 2 \), find \( \sec(-x) \).

63. If \( \csc x = -5 \), find \( \csc(-x) \).

64. If \( x\sin x = 2 \), find \( (-x)\sin(-x) \).

For the following exercises, rewrite each expression such that the argument \( x \) is positive.

65. \( \cot(-x)\cos(-x) + \sin(-x) \)

66. \( \cos(-x) + \tan(-x)\sin(-x) \)

**Graphical**

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

67. \( f(x) = 2\tan(4x - 32) \)

68. \( h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right) \)

69. \( m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right) \)

70. \( j(x) = \tan\left(\frac{\pi}{2}x\right) \)

71. \( p(x) = \tan(x - \frac{\pi}{2}) \)

72. \( f(x) = 4\tan(x) \)

73. \( f(x) = \tan\left(x + \frac{\pi}{4}\right) \)

74. \( f(x) = \pi\tan(\pi x - \pi) - \pi \)

75. \( f(x) = 2\csc(x) \)

76. \( f(x) = -\frac{1}{4}\csc(x) \)

77. \( f(x) = 4\sec(3x) \)

78. \( f(x) = -3\cot(2x) \)
79. \( f(x) = 7\sec(5x) \)

80. \( f(x) = \frac{9}{10}\csc(\pi x) \)

81. \( f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1 \)

82. \( f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2 \)

83. \( f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{3}\right) \)

84. \( f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right) \)

For the following exercises, find and graph two periods of the periodic function with the given stretching factor, \(|A|\), period, and phase shift.

85. A tangent curve, \( A = 1 \), period of \( \frac{\pi}{3} \), and phase shift \((h, k) = \left(\frac{\pi}{4}, 2\right)\)

86. A tangent curve, \( A = -2 \), period of \( \frac{\pi}{4} \), and phase shift \((h, k) = \left(-\frac{\pi}{4}, -2\right)\)

For the following exercises, find an equation for the graph of each function.

87. 

88.
89.

90.

91.
92.

Technology

For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input \( \csc x \) as \( \frac{1}{\sin x} \).

93.

94. \( f(x) = |\csc(x)| \)
95. \( f(x) = |\cot(x)| \)
96. \( f(x) = 2^{\csc(x)} \)
97. \( f(x) = \frac{\csc(x)}{\sec(x)} \)
98. Graph \( f(x) = 1 + \sec^2(x) - \tan^2(x) \). What is the function shown in the graph?
99. \( f(x) = \sec(0.001x) \)
100. \( f(x) = \cot(100\pi x) \)
101. \( f(x) = \sin^2 x + \cos^2 x \)
Real-World Applications

102. The function \( f(x) = 20 \tan \left( \frac{\pi}{10} x \right) \) marks the distance in the movement of a light beam from a police car across a wall for time \( x \), in seconds, and distance \( f(x) \), in feet.
   a. Graph on the interval \([0, 5]\).
   b. Find and interpret the stretching factor, period, and asymptote.
   c. Evaluate \( f(1) \) and \( f(2.5) \) and discuss the function’s values at those inputs.

103. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let \( x \), measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and \( x \) is measured negative to the left and positive to the right. (See Figure 6.53.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance \( d(x) \), in kilometers, from the fisherman to the boat is given by the function \( d(x) = 1.5 \sec(x) \).
   a. What is a reasonable domain for \( d(x) \)?
   b. Graph \( d(x) \) on this domain.
   c. Find and discuss the meaning of any vertical asymptotes on the graph of \( d(x) \).
   d. Calculate and interpret \( d \left( -\frac{\pi}{3} \right) \). Round to the second decimal place.
   e. Calculate and interpret \( d \left( \frac{\pi}{6} \right) \). Round to the second decimal place.
   f. What is the minimum distance between the fisherman and the boat? When does this occur?

104. A laser rangefinder is locked on a comet approaching Earth. The distance \( g(x) \), in kilometers, of the comet after \( x \) days, for \( x \) in the interval 0 to 30 days, is given by \( g(x) = 250,000 \csc \left( \frac{\pi}{30} x \right) \).
   a. Graph \( g(x) \) on the interval \([0, 35]\).
   b. Evaluate \( g(5) \) and interpret the information.
c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?

d. Find and discuss the meaning of any vertical asymptotes.

105. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after $x$ seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after $x$ seconds. Ignore the curvature of the Earth.

b. Graph $h(x)$ on the interval $(0, 60)$.

c. Evaluate and interpret the values $h(0)$ and $h(30)$.

d. What happens to the values of $h(x)$ as $x$ approaches 60 seconds? Interpret the meaning of this in terms of the problem.